Určitý integrál, aplikace

Opakování pojmů

- $\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \overset{\text{ozn.}}{=} [F(x)]_a^b$; $\quad a, b$ - dolní a horní mez
- $\int_a^a f(x)\mathrm{d}x = 0, \int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$
  pozor na nespojitost $\int_{-1}^1 \frac{1}{x}\mathrm{d}x$ nelze
- geometrický význam: obsah plochy pod křivkou
- vlastnosti

$$f(x) \leq g(x) \text{ na } <a, b> \Rightarrow \int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$$

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x, \text{ pro } c \in <a, b>$$

$$\int_a^b (Af(x) + Bg(x))\mathrm{d}x = A\int_a^b f(x)\mathrm{d}x + B\int_a^b g(x)\mathrm{d}x$$

$$\text{sudá funkce: } \int_{-a}^a f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$$

$$\text{lichá funkce: } \int_{-a}^a f(x)\mathrm{d}x = 0$$

$$f(x) \geq 0 \Rightarrow \int_a^b f(x)\mathrm{d}x \geq 0$$

- metoda po částech

$$\int_a^b u'(x)v(x)\mathrm{d}x = [u(x)v(x)]_a^b - \int_a^b u(x)v'(x)\mathrm{d}x$$

- metoda substituce: je třeba provést záměnu mezí podle zvolené substituce

Příklady

$$\int_1^3 \left(\frac{2}{\sqrt{x}} - \sqrt[3]{x}\right)\mathrm{d}x = \left[4\sqrt{x} - \frac{3}{4}\sqrt[3]{x^4}\right]_1^3 = \underline{\underline{4\sqrt{3} - \frac{3}{4}\sqrt[3]{81} - 4 + \frac{3}{4}}}$$

$$\int_1^{\sqrt{3}} \frac{\mathrm{d}x}{1+x^2} = [\mathrm{arctg}x]_1^{\sqrt{3}} = \mathrm{arctg}\sqrt{3} - \mathrm{arctg}1 = \frac{\pi}{3} - \frac{\pi}{4} = \underline{\underline{\frac{\pi}{12}}}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = [\arcsin x]_{-\frac{1}{2}}^{\frac{1}{2}} = \arcsin\frac{1}{2} - \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \underline{\underline{\frac{\pi}{3}}}$$

$$\int_{-1}^2 3x^2\mathrm{d}x = \ldots = 9, \qquad \int_1^8 \sqrt[3]{x}\mathrm{d}x = \ldots = \frac{45}{4}, \qquad \int_0^2 (3x^2 - 2x + 5)\mathrm{d}x = \ldots = 18$$

$$\int_1^4 \left(-x + \frac{4}{x}\right)\mathrm{d}x = \ldots = 4\ln 4 - \frac{15}{2}, \qquad \int_0^1 x^3(1-x)^2\mathrm{d}x = \ldots = \frac{1}{60}$$

$$\int_1^4 \sqrt{x}(1 + 2\sqrt{x})\mathrm{d}x = \ldots = \frac{59}{3}, \qquad \int_0^\pi 2\sin x\mathrm{d}x = \ldots = 4, \qquad \int_0^\pi \frac{1}{\cos^2 x}\mathrm{d}x = \ldots = 0$$

$$\int_0^{\frac{\pi}{4}} \frac{1 + \sin^2 x}{\cos^2 x}\mathrm{d}x = \ldots = 2 - \frac{\pi}{4}$$

Substituce

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x \mathrm{d}x = \begin{vmatrix} t = \sin x \\ dt = \cos x dx \end{vmatrix} = \int_0^1 (1-t^2)t^2 \mathrm{d}t = \left[\frac{1}{3}t^3 - \frac{1}{5}t^5\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \underline{\underline{\frac{2}{15}}}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x \cos x \mathrm{d}x = \ldots = \frac{7}{24}, \qquad \int_{-4}^{2} \sqrt{17+4x} \mathrm{d}x = \ldots = \frac{62}{3}$$

$$\int_0^1 \frac{x}{(x^2+1)^2} \mathrm{d}x = \ldots = \frac{1}{4}, \qquad \int_0^1 \frac{e^x}{e^x+1} \mathrm{d}x = \ldots = \ln(e+1) - \ln 2$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} \mathrm{d}x = \ldots = \ln 2$$

Per partes

$$\int_0^1 x2^x \mathrm{d}x = \begin{vmatrix} u' = 2^x & v = x \\ u = \frac{2^x}{\ln 2} & v' = 1 \end{vmatrix} = \left[\frac{2^x x}{\ln 2}\right]_0^1 - \int_0^1 \frac{2^x}{\ln 2} \mathrm{d}x = \frac{2}{\ln 2} - \frac{1}{\ln 2} \int_0^1 2^x \mathrm{d}x = \frac{2}{\ln 2} - \frac{1}{\ln 2} \left[\frac{2^x}{\ln 2}\right]_0^1 =$$

$$= \underline{\underline{\frac{2}{\ln 2} - \frac{1}{\ln^2 2}}}$$

$$\int_0^{\pi} x \sin x \mathrm{d}x = \ldots = \pi, \qquad \int_1^2 (3x+2) \ln x \mathrm{d}x = \ldots = 10 \ln 2 - \frac{17}{4}$$

$$\int_1^e x^3 \ln x \mathrm{d}x = \ldots = \frac{3e^4+1}{16}$$

Parciální zlomky

$$\int_1^2 \frac{\mathrm{d}x}{x+x^3} = \int_1^2 \frac{\mathrm{d}x}{x(1+x^2)} = \int_1^2 \frac{\mathrm{d}x}{x} - \int_1^2 \frac{x\mathrm{d}x}{1+x^2} = [\ln|x|]_1^2 - \frac{1}{2}\left[\ln|1+x^2|\right]_1^2 =$$

$$|A = 1, B = -1, C = 0| \qquad = \ln 2 - \frac{1}{2}\ln 5 + \frac{1}{2}\ln 2 = \ln 2\sqrt{2} - \ln\sqrt{5} = \underline{\underline{\ln 2\frac{\sqrt{2}}{\sqrt{5}}}}$$

Určete obsah S_M útvaru M ohraničeného zadanými křivkami, načrtněte grafy funkcí:

a) $y = x^2, y = \sqrt{x}$

$$S_M = \int_0^1 (\sqrt{x} - x^2) \mathrm{d}x = \left[\frac{2}{3}\sqrt{x^3} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \underline{\underline{\frac{1}{3}}}$$

b) $y = -x^2 + 2, y = 0$

$$S_M = 2\int_0^{\sqrt{2}} (-x^2+2) \mathrm{d}x = 2\left[-\frac{x^3}{3} + 2x\right]_0^{\sqrt{2}} = 2\left(-\frac{\sqrt{8}}{3} + 2\sqrt{2}\right) = 2\left(-\frac{2\sqrt{2}}{3} + 2\sqrt{2}\right) = \underline{\underline{\frac{8\sqrt{2}}{3}}}$$

c) $y = -x^2 + 2x + 8, y = 0$

$$S_M = \int_{-2}^4 (-x^2+2x+8) \mathrm{d}x = \ldots = 36$$

d) $y = 16 - x^2, y = x^2 - 16$

$$S_M = 2\int_0^4 (32 - 2x^2) \mathrm{d}x = \ldots = \frac{512}{3}$$

e) $y = -9 - x^2, y = -5x - 9$

$$S_M = \int_0^5 (-x^2 + 5x)\mathrm{d}x = \ldots = \frac{125}{6}$$

f) $y = x^2 + 1, y = 0, x = -1, x = 2$

$$S_M = \int_{-1}^2 (x^2 + 1)\mathrm{d}x = \ldots = 6$$